FA Analysis of measurements´ errors

1. Literature

W. Walcher, Praktikum der Physik (2. Aufl.), Kap. 1 und 2
W.H.H. Gränicher, Messung beendet – was nun?, B.G. Teubner Stuttgart 1994, kap. 1-3,9.

2.Motivation

The purpose of the training period is to gain experience in the evaluation of the results of measurements and also getting to know the measuring technology. In order to check the validity of a theoretical model, the quality and explanatory power of the measurement has to be known. Every physical measurement is subject to random variations and systematic divergences, which one summarizes in the concept of *error* (old name) or more suitable *measuring inaccuracy* (German Institute for Standardization DIN 1319) . In the experimental work one should learn in a very easy experiment which kinds of measuring inaccuracies appear and how to determine them; also how the inaccuracy of a single measurement affects the whole result.
In the following it is used the shorter name error, synonymous with measuring inaccuracy.

3. Tasks

From the list of measurements of the oscillation period of a thread(wire) pendulum the standard deviation and the accidental error of the average as well as the total errors are to be determined (for the oscillation period and the gravitational acceleration). It is **not** the purpose of the experiment to determine very exactly the gravitational acceleration, because there are more suitable measuring procedures for this.

4. Bases

Under the assumption of the approximation of the thread pendulum by a mathematical pendulum (material point, massless and inelastic thread) one gets the gravitational acceleration *g* from the oscillation period *t* and the thread length *l*

$$g = (2\pi)^2 \frac{l}{t^2} \quad . \qquad \text{(FA.1)}$$

The oscillation period *t* is given by the arithmetic average of *n* measurements $t_1, t_2, ..., t_n$ of the determined oscillation period:

$$\bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i \quad . \qquad \text{(FA.2)}$$

Assuming confidence interval of 95% (i.e. 5% of the measured values from an infinite series of measurements lie outside of the margins of error), the random deviation of the average value amounts to

$$\Delta \bar{t}_{zuf} = 2 \cdot s_{\bar{t}} \qquad \text{(with a confidence coefficient of 95\%) ,} \qquad \text{(FA.3)}$$

where the standard deviation $s_{\bar{t}}$ of the average value of *n* measured values is

$$s_{\bar{t}} = s_t \cdot \frac{1}{\sqrt{n}} \qquad \text{(FA4)}$$

and the standard deviation of the individual measured values is

$$s_t = \sqrt{\frac{1}{n-1}\sum\left(\bar{t}-t_i\right)} \ . \qquad \text{(FA5)}$$

If the measured values t_i correspond to a "GAUSS normal distribution", then the probability density of their occurrence is

$$\varphi(t) = \frac{1}{\sqrt{2\pi}\cdot s_t}\cdot \exp\left(-\frac{\left(\bar{t}-t\right)^2}{2{s_t}^2}\right) \ . \qquad \text{(FA6)}$$

Apart from the arandom error one has also to consider the systematic error of the chronometer. In our case the relative systematic error is given by the quartz accuracy, indicated by the manufacturer as $\Delta t_{sys} / t_{sys} = 0.01\%$. For the indirect measurement of the gravitational acceleration *g*, apart from the oscillation period *t* also the thread length *l* should be mentioned as measured variable. The practicants have to measure *l* with the help of a folding rule ("zollstock") (Think first about the place of the pendulum body from which you make the measurement of the length!). The random error Δl_{zuf} should be estimated; according to DIN the systematic error amounts to Δl_{sys}=200μm+5*$10^{-4}l$.
Along with the determination of the errors Δg_{sys} and Δg_{zuf}, one should pay attention to the different propagation of the systematic and random errors.
The random errors result from the Gauss law for propagation of errors:

$$\left|\Delta g_{zuf}\right| = \sqrt{\left(\frac{\delta g}{\delta t}\Delta t_{zuf}\right)^2 + \left(\frac{\delta g}{\delta l}\Delta l_{zuf}\right)^2} \ . \qquad \text{(FA7)}$$

When small changes Δt_{sys} and Δl_{sys} occure, then effect on the systematic errors of the independent variables *t* and *l* can be estimated by a first order Taylor expansion:

$$g\left(t+\Delta t_{sys}, l+\Delta l_{sys}\right) = g(t,l) + \frac{\delta g}{\delta t}\Delta t_{sys} + \frac{\delta g}{\delta t}\Delta l_{sys} + \ldots \qquad \text{(FA8)}$$

Thus one gets the contributin of the systematic errors:

$$\left|\Delta g_{sys}\right| = \left|\frac{\delta g}{\delta t}\Delta t_{sys}\right| + \left|\frac{\delta g}{\delta l}\cdot\Delta l_{sys}\right| \ . \qquad \text{(FA9)}$$

Systematic and random error should be indicated separately, e.g.
t = 3.2s ± 0.2s (*syst.*) ± 0.1s (*zuf.*) .
However, in practice one finds often indications of the overall errors.

5. Realisation of the experiment

The measurement of the oscillation period t of a mathematical pendulum takes place with the help of an electronic clock in the zero point of the motion. The individual results of measurements t_i are arranged in time intervals. To have a sufficient number of measurements in each interval, one determines the optimal number *r* of intervals from the length of the series of measurements *n*

$$r \approx \sqrt{n} \quad . \qquad \text{(FA10)}$$

From a preliminary test with about 10 measurements approximate values for $\bar{t}$ and s_t are calculated from the equations (FA.2) and (FA.4). From the approximate value for s_t and the necessity to be able to arrange at least 99,9% of the resulting measured values in these time intervals, it can be determined the width Δt_{Int} and the position of the intervals. From fig. FA.1 is easy to find the width of an interval

$$r \cdot \Delta t_{Int} = 6 \cdot s_t \quad .$$

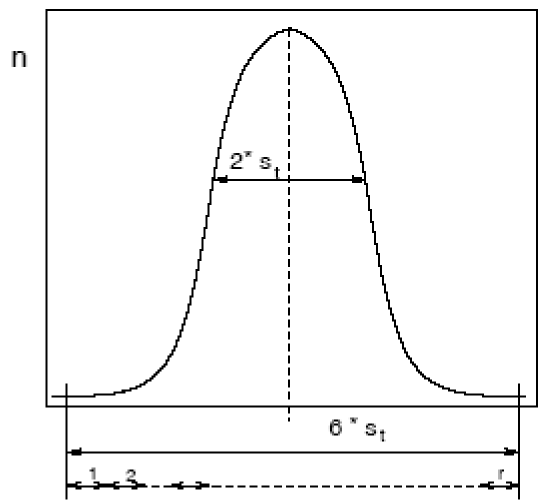

Abbildung FA.1: Schema zur Festlegung der Intervalle

The position of the individual intervals (the first intervals) has to be selected in such a way that the interval no.(*r/2*) contains the determined average value of the oscillation period $\bar{t}$ from the preliminary test. To be sure, one should add to this procedure at each side (before the interval 1 and after the interval r) another 2 to 3 intervals. The registration of the measured values into an interval list ("strichliste") should enable the evaluation (in form of a histogram) for n = 10, 25, 50, 100 and 200 determinations.

For the calculation of average value and standard deviation you can use your pocket calculator (if this has statistics functions) or your PC. In good approximation you can determine $\bar{t}$ and s_t also from the interval list, if you make use of the following estimations:

$$\sum_{i=1}^{n} t_i \approx \sum_{k=1}^{r} n_k \cdot t_k \text{ und}$$
$$\sum_{i=1}^{n} v_i^2 \approx \sum_{k=1}^{r} n_k \cdot v_k^2 \quad \text{(FA.12)}$$

(n_k - number of measured values in k-th interval, t_k – center of the interval, $v_k = t - t_k$)

Through the representation of the cumulative frequency

$$N_j = \frac{1}{n} \cdot \sum_{i=1}^{j} n_i \qquad \text{(FA.13)}$$

up to j-th interval according to the error integral

$$\phi(x) = \int_{-\infty}^{x} \varphi(t)dt \qquad \text{(FA.14)}$$

from the corresponding "probability diagram" (fig. FA.2) one can determine the quantity $\bar{t} \mp s_t$ from the points of ordinate of the relative cumulative frequency N_j=15,9% and 84.1%, respectively , while from N_j=50% the average value $\bar{t}$ can be read off. In addition one can test whether the measured values correspond to a series of measurements of a normal distribution.

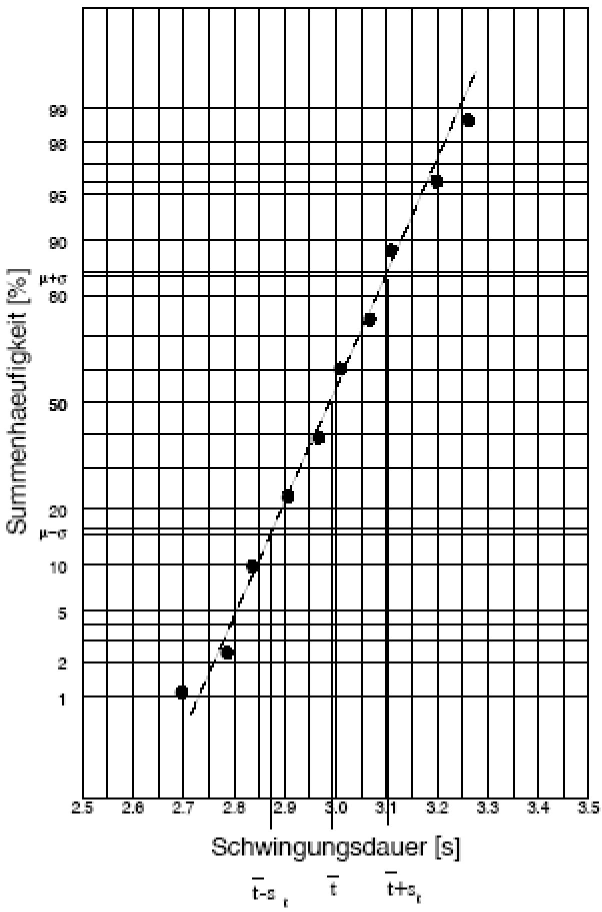

Abbildung FA.2: Summenhäufigkeit

6. Measurements

1) Measure the length of the pendulum from the suspension to the center of the pendulum weight. Note the systematic error Δl_{sys}. Estimate the random error Δl_{zuf}.
2) Measure 10 times the duration of a single oscillation of the pendulum
   - Take the time in the zero-point
   - move the pendulum *only little* (≈ 2 cm).

   Note the systematic error of time measurements.
3) Calculate immediately for these values $\bar{t}$ and s_t. Estimate a suitable width Δt for the intervals in your measuring table according to eq. (FA.11) with

$$\Delta t = \frac{6 \cdot s_t}{\sqrt{n}}, \qquad n = 200. \qquad \text{(FA.15)}$$

   Make resonble approximations. Write down an interval list with 6+√200 ~21 intervals, which are symmetrically grouped around $\bar{t}$, with separate columns for the first 10, 25, 50, 100 measurements:

Intervall [Sekunden]	Messw. 1 - 10	Messw. 11 - 35	Messw. 36 - 85	Messw. 86 - 185	Messw. 186 - 200
2.02 -					
2.04 -					
...					

4) Measure now the period duration of 200 single oscillations. Pay attention to register independently measured values (they are correlated with neighbouring values through sequential measurement!). Discuss this problem with the assistant and find together a suitable measuring procedure. The individual times should not be registered; it is sufficient to register lines into the Interval list.

7. Evaluation

1) Analyse the interval list. Draw a histogram for n= 10, 25, 50, 100, 200 measuring points. For n = 200 collect all columns.
2) Estimate in each case for n = 10, 25, 50, 100, 200 measuring points the standard deviation s_t, the average value $\bar{t}$ as well as the error of the average value $\Delta\bar{t}$ according to eqs. (FA.2-FA.5).
3) Calculate the gravitational acceleration g using the average value $\bar{t}$ which you got from the 200 measurements. Calculate Δg by using the errors for l and t and as indicated in eqs. (FA.7-FA.9). Pay attention to the different computations of systematic and random error and write down these errors separetely.
4) Register the distribution for n=200 into the probability net and from here read off $\bar{t}$ and s_t (line of best fit).

8. Questions to bases and investigational procedure

- What is a mathematical pendulum? Indicate the equation of motion and calculate the oscillation period!
- Why with this experiment is performed the determination of the oscillation period at zero-point oscillation and not at the point of reversal? When is it, depending on the measured oscillation period, more favorable for reaching small random errors: to stop in the point of reversal or in zero-point?
- For what reason should the measurements of the oscillation period made only by one student?
- How large should one select the width of the intervals, if for the measurements of the oscillation period (n = 200) a standard deviation of $s_t = 0.2s$ results?
- How does the random error change, if one determines the oscillation period from the measurement of 200 times one oscillation period instead of determining it from a single measurement of the duration of 200 oscillations?

9. Example

Indicated are: n (Entries), $\bar{t}$ (Mean), $\sqrt{\overline{t^2} - \bar{t}^2} = \sqrt{\frac{n-1}{n}} s_t$ (RMS, roots mean square deviation).

A Gauss function was adapted $n_k(t) = n_0 \cdot \exp\left(-\frac{\left(t-\bar{t}\right)^2}{2s_t^{\ 2}}\right)$. The sizes of Constant, Mean and Sigma correspond thereby to n_0, $\bar{t}$, s_t, respectively.

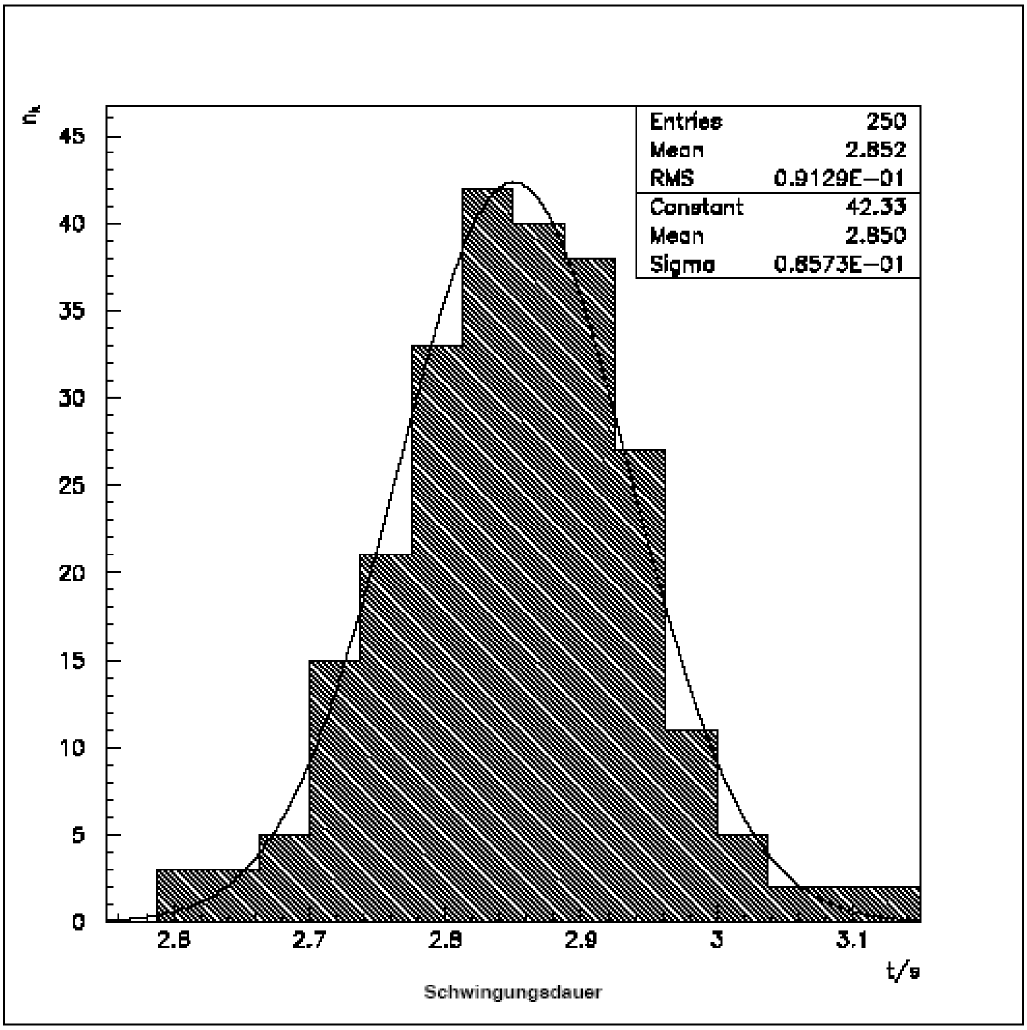

Illustration FA.3: Example of a frequency histogram for n=250 measured values.