The photon transfer function for accretion disks around a Kerr black hole

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Abstract

The observed spectrum from a thin accretion disk around a black hole can be calculated with the help of a transfer function $f$ provided the local emission in the disk plane is given. This function contains the Doppler boost due to the disk rotation, the relativistic light deflection, and the gravitational redshift. In this paper the transfer function is calculated and some of its properties are discussed. A fast and easy-to-use computer program is presented to obtain numerical values of $f$ for any set of parameters.

1. Introduction

The standard model for the source of radiation from active galactic nuclei (AGN) is accretion onto a massive black hole (Rees [10]), and if the accreted matter carries sufficient angular momentum an accretion disk will form around the central object. The local emission properties of such accretion disks can be obtained by detailed model calculations of the disk structure including radiative transfer. Once the emitted spectrum from every point on the disk surface is known the total spectrum is obtained by integrating over the entire surface taking into account the Doppler boost due to the disk rotation as well as the relativistic light deflection in the vicinity of the central black hole. All these effects can be cast into a transfer function which operates as an integration kernel to calculate the overall disk spectrum from the locally emitted radiation. This transfer function has been presented by C.T. Cunningham [5] who, however, published only a small table and a few figures for selected sets of parameters. Nevertheless, since then these values have been widely used to analyze observational data of AGN.

In this paper we present a computer program for calculating the transfer function using a slightly different method than Cunningham. The basic model consists of a thin rotationally symmetric stationary accretion disk with a given dependence of the gas velocity on the radial coordinate $r$, and a Kerr black hole of given mass $M$ and angular momentum $a$ viewed by a distant observer at position $r_o$ under an inclination angle $\theta_o$ (see Fig. 1). Photons are emitted from every point on the disk surface, and the radiation field depends on the frequency and the propagation direction which is generally specified by two angles. However, due to the rotational symmetry of the entire model the emitted radiation will only be a function of the propagation angle with respect to the disk normal but not of the corresponding azimuthal angle.
This basic physical model is briefly described in Section 2, while some more technical aspects like the calculation of the null geodesics and the gas velocity law are given in the Appendices A and B. In Section 3 the definition of the transfer function, its calculation, and some of its properties are discussed in detail. The usage of the computer program for calculating this function is described in Section 4, and finally a brief comparison with some of Cunningham’s results is presented.

The computer program is strictly written in FORTRAN 77. It consists of several subroutines which can be called from a user supplied main program. These subroutines are available from the authors through electronic mail.

2. The basic model

In the standard thin accretion disk model the gasdynamics is dominated by the gravity of the central mass, and the gas moves approximately on circular geodesic orbits in the equatorial plane of the supermassive rotating black hole, with a small velocity component towards the center. This inward mass flow is caused by viscous processes in the disk which at the same time lead to an outward angular momentum flux. The gravitational energy gained by this inflow is locally radiated away. This radiation, moving on null geodesics outside the disk, is then seen by the distant observer. The gas- and photon dynamics of the entire system can be calculated using a given background Kerr metric [3,6,8]. The extension of the disk in the vertical direction can be neglected compared with its radial extension. For the circular orbits there exists a radius of marginal stability \( r_{\text{ms}} \), and inside this radius the gas falls on geodesics with constant energy and angular momentum through the event horizon \( r_+ \) into the black hole. (For a more detailed description of the accretion disk model see Appendix B).

The local spectrum of the radiation at the surface of the disk is usually described in terms of the specific intensity \( I_{\nu_e} \), and has to be obtained from some detailed disk structure calculation which is considered as a given input model for the problem discussed in this paper. Therefore, the starting point is the intensity \( I_{\nu_e} \) being a function of the emission radius \( r_e \), the frequency \( \nu_e \), and an emission angle \( n_e \) measured with respect to the local disk normal (see Fig. 1). The intensity is assumed to be rotationally symmetric, thus effects depending on the azimuthal angle of the radiation can not be handled by this model.

On its way to the distant observer the spectrum is changed by Doppler shifts because of the high velocity of the emitting gas in the rotating disk. In addition, it is modified by the gravitational redshift, and the photons are deflected in the gravitational field of the black hole. These relativistic effects on the radiation are demonstrated in Fig. 2, where a visualization of the accretion disk is shown, i.e. the distortion of the disk a distant observer would see. The geometry of the disk is indicated by concentric circles around the black hole with equal differences in the radial coordinate. In the Newtonian case these circles would have the shape of ellipses. However, due to gravitational light deflection the ellipses are strongly distorted. Furthermore, the photons from

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Fig. 2. Relativistic effects in the vicinity of an accretion disk seen by a distant observer with \( \cos(\theta_o) = 0.25 \); in the left figure \( a = 0.9981 M \) (see [12]), and in the right one \( a = 0 \) (Schwarzschild case). Presented is the distortion of concentric circles in the equatorial plane of the black hole. The lines of equal redshift \( g \) are also indicated on the surface of the disk.

the disk are shifted in frequency depending on the point of emission. Close to the event horizon, where the rotation velocity is largest, the photons are extremely shifted in energy; Fig. 2 shows a photon redshift on the right hand side of the black hole, and a blueshift on the other side where the gas in the disk is moving towards the observer. With these geometric informations of the relativistic effects and with a given specific intensity \( I_{\nu_c} \) it would be easy to present an image of the disk, i.e. the spatial distribution of the specific intensity.

In all the following calculations, the occurrence of higher order images, i.e. null geodesics which are deflected in such a way that they circle the black hole once or many times before reaching the distant observer, is neglected. The disk is therefore considered to be opaque up to the event horizon, and its radial extension is assumed to be large enough that no photon from the hidden side of the disk can reach the observer. For large inclination angles \( \theta_o \), the latter assumption is not very realistic because for \( \theta_o \to \pi/2 \) the radial extension of the disk will tend to infinity, and for \( \theta_o = \pi/2 \) light from both sides of the disk will always be seen. This has to be taken into consideration when using large \( \theta_o \).

3. The transfer function

3.1. Definition of the transfer function

The observed specific flux \( F_{\text{obs}}^{(\nu)} \) is defined as

\[
F_{\text{obs}}^{(\nu)} = \int I_{\nu_o} \cos(\theta) \, d\Omega,
\]

where the integration is performed over the solid angle \( \Omega \) related to the observer, and where \( I_{\nu_o} \) is the observed specific intensity and \( \theta \) is the angle between the direction to the observed object and the direction of the photon propagation. Here the observer is assumed to be far away from the emitting accretion disk, therefore \( \cos(\theta) \approx 1 \). According to Liouville’s theorem the observed specific intensity \( I_{\nu_o} \) is related to the emitted specific intensity \( I_{\nu_e} \)

\[
\frac{I_{\nu_o}}{\nu_o^3} = \frac{I_{\nu_e}}{\nu_e^3}
\]

for freely propagating photons (Lindquist [7]), where \( \nu_o \) and \( \nu_e \) are the observed and emitted frequencies, respectively. The ratio

\[
g \equiv \frac{\nu_o}{\nu_e}
\]
is an important measure of the photon redshift along their geodesics. Thus, the observed specific flux is related to the emitted intensity

\[ F_{\text{obs}}^{(\nu)} = \int g^3 I_{e}\,d\Omega. \]  

\( I_{\nu} = I_{\nu}(\nu_e, r_e, n_e) \) is considered to be a given function which has to be calculated from a detailed accretion disk model. It depends only on the emitted frequency \( \nu_e \), the emission radius \( r_e \), and on the direction of photon emission. The latter is described by the polar angle \( n_e \) of the emitted photon with respect to the disk normal in the local rest frame of the disk; this includes the assumption of a rotationally symmetric intensity at every emission point, otherwise an additional (azimuthal) angle would be required to specify the propagation direction. It is convenient to express the solid angle \( \Omega \) in terms of the emission radius \( r_e \) and the relative redshift \( g^* \),

\[ g^* = \frac{g - g_{\text{min}}}{g_{\text{max}} - g_{\text{min}}}, \quad 0 \leq g^* \leq 1. \]  

\( g_{\text{max}} = g_{\text{max}}(r_e, \theta_o) \) and \( g_{\text{min}} = g_{\text{min}}(r_e, \theta_o) \) are the maximum and minimum values of \( g \) for all photon trajectories extending from the radius \( r_e \) to the distant observer located at position \( \theta_o \) (see Fig. 1). We now introduce the transfer function \( f \),

\[ f(g^*, r_e, \theta_o) = \frac{\rho_e}{\pi r_e} g \sqrt{g^*(1 - g^*)} \left| \frac{\partial \Omega}{\partial (g^*, r_e)} \right|, \]  

which was first defined by Cunningham [5]. With this transfer function, the specific flux (4) reads

\[ F_{\text{obs}}^{(\nu)} = \frac{1}{r_o^2} \int_0^1 \int r_e \frac{g^2}{\sqrt{g^*(1 - g^*)}} f(g^*, r_e, \theta_o) I_{\nu}(\nu_e, r_e, n_e) \,dg^*dr_e. \]  

This integral has to be calculated for a given set of parameters \( \theta_o, \nu_o, \alpha, \) and \( M \). The two arguments \( \nu_e = \nu_o/g \) and \( n_e \) of the intensity \( I_{\nu} \) have to be expressed in terms of \( r_e \) and \( g^* \), thus the two functions \( g_{\text{max}} = g_{\text{max}}(r_e, \theta_o) \) and \( g_{\text{min}} = g_{\text{min}}(r_e, \theta_o) \) are required to calculate

\[ g = (g_{\text{max}} - g_{\text{min}}) g^* + g_{\text{min}}. \]  

The parameter \( g^* \) on an emitting ring of radius \( r_e \) is double-valued, therefore, there exist two branches of the transfer function \( f_1(g^*, r_e, \theta_o) \) and \( f_2(g^*, r_e, \theta_o) \) (and correspondingly for the angle \( n_e \)). The points given by \( g^* = 1 \) and \( g^* = 0 \) (i.e. \( g = g_{\text{max}} \) and \( g = g_{\text{min}} \)) divide the ring into two parts, and on each section \( g^* \) ranges from 0 to 1. An exceptional situation occurs for \( \theta_o = 0 \) (observer on the rotational axis of the black hole) and for \( \theta_o = \pi/2 \) (observer in the equatorial plane). Thus the integral (7) over \( g^* \) has to be performed twice, first using \( f_1 \) and \( n_1 \), and then \( f_2 \) and \( n_2 \). If, however, the emission is isotropic in the rest frame \( I_{\nu} \) does not depend on \( n_e \) and is the same on both branches for a given value of \( g^* \). Then the corresponding functions \( f_1 \) and \( f_2 \) can be added and only one integration over \( g^* \) with \( f(g^*) = f_1(g^*) + f_2(g^*) \) is required.

3.2. Calculation of the transfer function

To calculate the transfer function (6) for given values \( r_e \) and \( g^* \) (i.e. for a pair of null geodesics connecting the emission ring with the observer) the Jacobian has to be computed after calculating \( g_{\text{max}}(r_e, \theta_o) \) and \( g_{\text{min}}(r_e, \theta_o) \). The impact parameters \( \alpha \) and \( \beta \) of these geodesics are related to their constants of motion \( \lambda \) and \( q \) (see Appendix A) by

\[ \alpha = -\frac{\lambda}{\sin(\theta_o)} \quad \text{and} \quad \beta = \pm \sqrt{V_0(\theta_o)} \]
(with $V_\theta(\theta_o)$ from (A.4)) for a distant observer (Cunningham, Bardeen [4]). Therefore

\[ r_o^2 d\Omega = d\alpha d\beta = \left| \frac{\partial(\alpha, \beta)}{\partial(\lambda, q)} \right| d\lambda dq = \frac{q}{\sin(\theta_o)\beta} d\lambda dq, \]  

and the Jacobian in (6) can be expressed in terms of the variables $\lambda$ and $q$,

\[ r_o^2 \left| \frac{\partial \Omega}{\partial (g^*, r_e)} \right| = \frac{q(g_{\max} - g_{\min})}{\sin(\theta_o)\beta} \left| \frac{\partial(\lambda, q)}{\partial (g, r_e)} \right|. \]  

From the momentum $p_e$ of the null geodesic at the emission point (see (A.2) in Appendix A) and the 4-velocity $u$ of the emitting gas (see (B.3), (B.4) in Appendix B) the redshift $g$ can be calculated,

\[ g = \frac{v_o}{v_e} = \frac{E_o}{E_e} = -\frac{E}{p_e u}. \]  

This leads to a rather complicated equation for $g$, $r_e$, $\lambda$ and $q$, different for the two cases $r_e \geq r_{ms}$ and $r_e < r_{ms}$. But in both cases this equation can be solved analytically with respect to $\lambda$, and we obtain a function $\lambda = \lambda(g, r_e, q)$. This function $\lambda$ can be inserted into the equation of motion (A.5) for the null geodesics. Now we have an implicit equation for $q = q(g, r_e)$ with the free parameters $\theta_o$, $a$ and $M$. This equation has to be solved numerically for the different cases discussed in Appendix A. For $g_{\min} < g < g_{\max}$ there are two solutions, for $g = g_{\min}$ or $g = g_{\max}$ there is only one solution and in all other cases no solution exists at all. Thus, by varying $g$ one can determine the extreme values of the redshift by the number of solutions. These calculations lead to the two functions $\lambda(g, r_e)$ and $q(g, r_e)$. Now the Jacobian on the right hand side of Eq. (11) can be computed numerically using finite differences,

\[ \left| \frac{\partial(\lambda, q)}{\partial (g, r_e)} \right| = \left| \frac{\partial \lambda}{\partial g} \frac{\partial q}{\partial r_e} - \frac{\partial q}{\partial g} \frac{\partial \lambda}{\partial r_e} \right|, \]  

which completes the calculation of the transfer function $f$.

Cunningham used a different way in his work to calculate the Jacobian in Eq. (11). He expressed $g$ and $r_e$ in terms of $\lambda$ and $q$ and obtained the inverse Jacobian

\[ \left| \frac{\partial (g, r_e)}{\partial (\lambda, q)} \right| = \left| \frac{\partial (\lambda, q)}{\partial (g, r_e)} \right|^{-1}. \]  

The main disadvantage of this method is that usually not the constants of motion $\lambda$ and $q$ are given but $g$ and $r_e$, and therefore a further costly inversion has to be performed.

In addition to $f$, the emission angle $n_e$ can be calculated easily,

\[ \cos(n_e) = -\frac{n_p_e}{u p_e} = \frac{q g}{r_e} \]  

where $n$ is the surface normal (B.1).

4. The program

4.1. Description of the program and its usage

The authors have written a program to perform all the calculations presented in the previous section. It is written strictly in FORTRAN 77 according to the standard to guarantee compatibility and consists of 25 subroutines with together more than 2200 lines of code, where several standard algorithms are used, e.g.
Gaussian quadratures etc. (Press et al. [9], or [11]). However only 3 of these 25 subroutines are important for the user, because they have to be included in his main program (the number 3 is due to economic reasons concerning the CPU time; see below). All other subroutines are only called internally. It is important to compile all routines with double precision to achieve the required accuracy.

The user has to write a main program to calculate the quantities he is interested in, e.g. the specific flux $F_{\text{obs}}^{(r)}$ or the specific luminosity $L_{\nu_0} = 4\pi r_0^2 F_{\text{obs}}^{(r)}$ (a sample main program is included in the Appendix C). First he has to define the parameters of the black hole and of the observation, i.e. the specific angular momentum $a$ of the black hole, the observation angle $\theta_0$, and the frequency $\nu_0$. The scaling of the radial coordinate in terms of $GM/c^2$ is used to eliminate the black hole mass. With these values he has to call the subroutine \textsc{DEFPAR} to make these parameters known to all subroutines and to define several other parameters. Whenever $a$ or $\theta_0$ change their values during runtime, \textsc{DEFPAR} has to be called again.

To compute a formula like the specific flux (7) which has the form

$$F_{\nu_0} = \int_{r_0}^{\infty} \int_0^1 h(g^*, g, r_e) f(g^*, r_e, \theta_0) I_{\nu_e}(\nu_e, r_e, n_e) \, dg^* \, dr_e,$$

where

$$h(g^*, g, r_e) = \frac{\pi r_e}{r_0^2} \frac{g^2}{\sqrt{g^*(1 - g^*)}},$$

one has to perform a numerical integration. The infinite integral over $r_e$ can be transformed onto a finite interval through the substitution $r_e = 1/x$. Using an appropriate quadrature formula we obtain

$$F_{\nu_0} = \sum_{i=0}^{N_{x_0}} w_i \sum_{j=0}^{N_{x_0}} \hat{w}_j h^{ij} f^{ij} I_{\nu_e}^{ij}(\nu_e^{ij}, n_e^{ij}).$$

(15)

The superscripts $i$ and $j$ denote the function values at the discrete abscissae $r_{ei}$, $g_{ij}^*$; $w_i$ and $\hat{w}_j$ stand for the corresponding integration weights. These quantities have to be specified in the user supplied main program. For each value $r_{ei}$ the extreme values $g_{\text{max}}$ and $g_{\text{min}}$ are computed in the subroutine \textsc{GEXTRM}. Because this calculation needs a significant fraction of the total CPU time, it is performed in a separate subroutine and its results are used as input for the subroutine \textsc{TRFFCT} which returns the transfer function. Now, the value of the redshift can be calculated

$$g^{ij} = g(g^{ij}, r_{ei}) = (g_{\text{max}} - g_{\text{min}}) g_j^* + g_{\text{min}},$$

and with all these parameters the subroutine \textsc{TRFFCT} computes the values of the transfer functions $f_1^{ij}$ and $f_2^{ij}$, together with the emission angles $n_{ei}^{ij}$ and $n_{ej}^{ij}$. Next, the values of the specific intensity $I_{\nu_e}^{ij}(\nu_e^{ij}, n_e^{ij})$ have to be supplied by a user defined sub-program which returns the results from an accretion disk model calculation. The only parameter still needed is $\nu_e^{ij}$ which is, however, easily obtained from $\nu_e^{ij} = \nu_0 / g^{ij}$ where $\nu_0$ is the chosen observed frequency.

We can now summarize the above computational procedure: First the global model parameters $a, \theta_0, \nu_0$ have to be specified. Next one needs to choose a quadrature formula, set up the grids $r_{ei}$ and $g_{ij}^*$, and calculate the corresponding weights $w_i$ and $\hat{w}_j$. Calling our program then delivers the values of $g_{\text{max}}, g_{\text{min}}, f, n_e$, and $r_e$ at the grid points. The final step consists of a call to some disk model routine to obtain the intensity $I_{\nu_e}^{ij}(\nu_e^{ij}, n_e^{ij})$ and perform the double sum in Eq. (15).

Usually the values of $f$ and $n_e$ are needed for a large number of pairs $(g^*, r_e)$. By a suitable order of calculation one can save some CPU time because the computation of the extreme values of $g$ for a given
Fig. 3. The dependence of the transfer function \( f(g^*, r_e, \theta_0) \) on the redshift \( g^* \) for \( a = 0.9981M \) and \( r_e = 4M \). Curves are labeled with the value of \( \cos(\theta_0) \). On the left hand side are the graphs given by Cunningham and on the right hand side are values computed by the authors.

\( r_e \) needs much more time than the computation of a single value of the transfer function. Thus, it is more reasonable to specify \( r_e \) first and then calculate \( f \) for different values of \( g \) than the other way round, because for each new value of \( r_e \) the subroutine GEXTRM has to be called. A call of GEXTRM requires about 10 sec. CPU time on an average workstation whereas a call of TRFFCT needs about 0.6 sec.

4.2. Results and discussion

We have performed a number of comparisons between the results presented in Cunningham’s paper [5] and the values of \( f, g_{\text{max}}, g_{\text{min}}, \) and \( n_e \) calculated with our program. Because Cunningham has published only a few tabulated values and some graphs (see Fig. 3), the comparisons are more or less qualitative. But within the accuracy given in [5] (values are only tabulated with 3 digits) the results are almost exactly identical. In Fig. 3 the transfer function \( f \) is shown for special parameters \( (a = 0.9981M, r_e = 4M \) and several values of \( \cos(\theta_0) \)). The graphs on the left hand side are taken from the paper [5], the graphs on the right hand side are computed by the method of the authors. There are no visual differences.

With the help of the program a relatively fast and easy calculation of the transfer function is possible for arbitrary sets of parameters, and with at least the same accuracy as the values given in [5]. In addition, a kind of visualization of the accretion disk is possible. The pictures shown in Fig. 2 are computed with some of the routines (mainly the one to calculate the impact parameters \( \alpha \) and \( \beta \)) which were only slightly modified.

Appendix A. Notes on the Kerr metric

In this section, some results of the Kerr metric are given, e.g. the equations of motion for the null geodesics, without any proof.

In Boyer-Lindquist coordinates the line element of the Kerr metric is

\[
 ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\varphi - \omega dt)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2. 
\] (A.1)
with
\[ e^{2\nu} = \frac{\Sigma}{A}, \quad e^{2\phi} = \sin^2(\vartheta) \frac{A}{\Sigma}, \quad \omega = 2Ma/A, \]

and
\[ \Delta \equiv r^2 - 2Mr + a^2, \quad \Sigma \equiv r^2 + a^2 \cos^2(\vartheta), \quad A \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2(\vartheta). \]

The parameter \( M \) represents the mass of the Kerr black hole and the parameter \( a \) the specific angular momentum, i.e. the angular momentum per unit mass. The event horizon lies at \( r_+ = M + \sqrt{M^2 - a^2} \).

For a geodesic the four constants of motion are:

- \( E \) : the energy at infinity,
- \( L_z \) : the angular momentum,
- \( \mu \) : the rest mass, and
- \( Q \) : (Carter [2]),

or \( E, \mu, \lambda = L_z/E \) and \( q^2 = Q/E^2 \). With these constants the momentum of a geodesic has the components

\[ p_r = -E, \quad p_\vartheta = \pm \frac{E}{\Delta} \sqrt{V_r}, \quad p_\varphi = \pm E \sqrt{V_\vartheta}, \quad p_\vartheta = E\lambda, \quad (A.2) \]

with

\[ V_r \equiv r^4 + (a^2 - \lambda^2 - q^2) r^2 + 2M ((a - \lambda)^2 + q^2) r - a^2q^2 - \frac{\mu^2}{E^2} \Delta, \quad (A.3) \]

\[ V_\vartheta = q^2 - \cos^2(\vartheta) \left( \frac{\lambda^2}{\sin^2(\vartheta)} - a^2 + a^2 \frac{\mu^2}{E^2} \right). \quad (A.4) \]

Integrating these equations with \( \mu = 0 \) gives the equations of motion for the null geodesics. Because of the axisymmetry and the steady-state assumptions for the accretion disk, only the coordinates \( r \) and \( \vartheta \) have to be considered. This leads to

\[ \int_{r_c}^{\infty} \frac{dr}{\pm \sqrt{V_r}} = \int_{\vartheta_0}^{\pi/2} \frac{d\vartheta}{\pm \sqrt{V_\vartheta}}. \quad (A.5) \]

The radial component of the null geodesic reaches from the emission radius \( r_c \) to the distant observer at infinity, and the \( \vartheta \) component reaches from the equatorial plane \( (\vartheta = \frac{\pi}{2}) \) to the observation angle \( \vartheta_0 \).

The different signs indicate the direction of the motion for each coordinate. The signs can only be changed if the geodesic passes a turning point of the corresponding coordinate (i.e. \( V_r = 0 \) or \( V_\vartheta = 0 \)). For the radial integration there are only two possibilities

\[ \int_{r_c}^{r_{null}} \frac{dr}{\sqrt{V_r}} \quad \text{and} \quad - \int_{r_{null}}^{r_{null}} \frac{dr}{\sqrt{V_r}} + \int_{r_{null}}^{\infty} \frac{dr}{\sqrt{V_r}}. \]

In the first case the photon is straight going outwards. In the second case the photon is initially going inwards toward the black hole until it reaches the turning point \( r_{null} \) and then goes outwards to the distant observer. Note that the photon can never leave the black hole if \( r_{null} \leq r_+ \).
Similarly, for the integration over \( \vartheta \) there are also only two possibilities for the motion if we consider the accretion disk as being opaque. With the substitution \( \eta = \cos(\vartheta) \) we have
\[
\int_{0}^{\cos(\vartheta_{e})} \frac{d\eta}{\sqrt{V_{\eta}}} \quad \text{and} \quad \int_{0}^{\sqrt{\eta_{\max}}} \frac{d\eta}{\sqrt{V_{\eta}}} - \int_{\sqrt{\eta_{\max}}}^{\cos(\vartheta_{e})} \frac{d\eta}{\sqrt{V_{\eta}}},
\]
where \( V_{\eta} = -a^{2}\eta^{4} + (a^{2} - \lambda^{2} - q^{2}) \eta^{2} + q^{2} \) and \( \sqrt{\eta_{\max}} \) is the maximum root of \( V_{\eta} \).

**Appendix B. Model of the accretion disk**

In this section, the accretion disk is described more mathematically and some important formulas are given. The disk is located in the equatorial plane of the rotating black hole, it is axisymmetric, in a steady state, and it is thin, i.e., its extension in the vertical direction can be neglected; for the subsequent calculations we assume \( \vartheta = \pi/2 \). Thus, its surface normal in the rest frame reads
\[
n = \left. \frac{1}{\sqrt{\Sigma}} \frac{\partial}{\partial \vartheta} \right|_{\vartheta=\pi/2}.
\] (B.1)

The gas in the disk flows very nearly on circular orbits, slowly spiraling inwards until it reaches the radius of marginal stability \( r_{\text{ms}} \) (Bardeen et al. [1]):
\[
r_{\text{ms}} = M \left( 3 + Z_{2} - \sqrt{(3 - Z_{1})(3 + Z_{1} + 2Z_{2})} \right),
\] (B.2)
\[
Z_{1} := 1 + \left( 1 - \frac{a^{2}}{M^{2}} \right)^{1/3} \left( \left( 1 + \frac{a}{M} \right)^{1/3} + \left( 1 - \frac{a}{M} \right)^{1/3} \right), \quad Z_{2} := \sqrt{\frac{3a^{2}}{M^{2}} + Z_{1}^{2}}.
\]

Outside \( r_{\text{ms}} \), the gas has the 4-velocity
\[
u = u^{\prime} \left( \frac{\partial}{\partial t} + \Omega_{\varphi} \frac{\partial}{\partial \varphi} \right),
\] (B.3)
with
\[
\Omega_{\varphi} = \frac{\sqrt{M}}{r \sqrt{r + a \sqrt{M}}} \quad \text{and} \quad u^{\prime} = \frac{r \sqrt{r + a \sqrt{M}}}{\sqrt{r \sqrt{r^{2} - 3Mr + 2a \sqrt{M} \sqrt{r}}}}.
\]

At \( r = r_{\text{ms}} \) the circular geodesics become unstable and the gas falls into the black hole with energy \( E_{e} \) and angular momentum \( L_{e} \) of the geodesic of marginal stability. For \( r \leq r_{\text{ms}} \) the 4-velocity of the gas is given by
\[
u = u^{\prime} \frac{\partial}{\partial t} + u^{\prime} \frac{\partial}{\partial r} + u^{\varphi} \frac{\partial}{\partial \varphi},
\] (B.4)
\[
u^{\prime} = \frac{E_{e}}{\mu} \left( 1 + \frac{2M}{r} \left( 1 + H \right) \right), \quad u^{\varphi} = -\sqrt{\frac{2M}{3r_{\text{ms}}}} \left( \frac{r_{\text{ms}}}{r} - 1 \right)^{2/3}, \quad u^{r} = \frac{1}{r^{2}} \frac{E_{e}}{\mu} \left( \frac{L_{e}}{E_{e}} + aH \right)
\]
with
\[
H = \frac{1}{\Delta} \left( 2Mr - a \frac{L_{e}}{E_{e}} \right), \quad \frac{E_{e}}{\mu} = \sqrt{1 - \frac{2M}{3r_{\text{ms}}}}.
\]
\[
\frac{L_c}{E_c} = \frac{\sqrt{M (r_{ms}^2 - 2a \sqrt{M r_{ms}^2 + a^2})}}{r_{ms} \sqrt{r_{ms}^2 - 2M r_{ms}^2 + a^2}}.
\]

Appendix C. Sample main program

Sample main program to calculate the specific luminosity \( L_{\nu_o} \) (see Section 4.1):

C Sample main program to calculate the specific luminosity \( L_{\nu_o} \) for
C given values of the angular momentum \( a \), observation angle \( \theta_{0} \), and
C various observed frequencies \( \nu_o \). All radii are given in units of
C GM/c**2 where \( M \) is the mass of the black hole.
C This program is an example of how to use the subroutines
C DEFPAR, GEXTRM, and TRFFCT.
C The integrations are carried out by Gauss-Legendre-quadratures. The
C abscissas and weights are computed in the subroutine GAULEG, which
C is part of the subroutines of GEXTRM and TRFFCT.
C The specific intensity \( I_{\nu_{0}} \) should be computed in the function
C FCTISP. This routine has to be supplied by the user.
C The program stops if the input of \( \nu_o \) is less than 0.

PROGRAM SAMPLEMAIN

C-------------------- Declaration of variables ---------------------
REAL*8 PI,A,THETAO,RMS,RPLUS,NVG,NUE,LSPEC
INTEGER NRE,NG,I,J,K

C Number of abscissas for integration with respect to \( r_e \) and \( g^* \):
PARAMETER (NRE = 10 , NG = 10)

REAL*8 R1,R2,R(NRE),RE(NRE),WR(NRE),INTRL(2),
& G1,G2,STG(NG),G(NRE,NG),W0(NG),INTGL(2),INTG
& GMIN(NRE),GMAX(NRE),TRFF(NRE,NG,2),COSNE(NRE,NG,2)
& TRFFH(2),COSNEH(2),ISPEC(NRE,NG,2)

EXTERNAL DEFPAR,GEXTRM,TRFFCT,GAULEG,FCTISP

C------------------ Initialisations -----------------------------
C Set constants:
\( \pi = 4.0 \times ATAN(1.0) \)
C Angular momentum \( a \):
\( a = 0.998190 \)
C Input of observation angle \( \theta_0 \):
READ *,THETAO
C Initialize \( a \) and \( \theta_0 \):
CALL DEFPAR(A,THETAO,RMS,RPLUS)

C------------------- (\( r_e, g^* \))-grid of integration ----------
C Range of integration with respect to \( r_e \):
R1 = 0.0
C Radiation for \( r_e > r_{\text{plus}} \) (from horizon):
R2 = 1.0 / RPLUS

C- Radiation for re > rms (stable disk only):
   R2 = 1.0 DO / RMS

C- Abscissas and weights for re-integration:
   CALL GAULEG(R1,R2,R,WR,NRE)
C- Corresponding emission radius re:
   DO 10 I = 1, NRE
      RE(I) = 1.0 DO / R(I)
   10 CONTINUE

C- Range of integration with respect to g*:
   G1 = 0.0 DO
   G2 = 1.0 DO
C- Abscissas and weights for g*-integration:
   CALL GAULEG(G1,G2,GSTR,WG,NG)

C-----------------------------------------------

C---------- Values of the transfer function f ------
   DO 20 I = 1, NRE
C- Calculation of the corresponding extreme values of the redshift:
   CALL GEXTRM(RE(I),GMIN(I),GMAX(I))
   DO 30 J = 1, NG
C- Calculation of the redshift at gridpoint (re,g*)(i,j):
   G(I,J) = (GMAX(I) - GMIN(I))*GSTR(J) + GMIN(I)

C- Calculation of the transfer function and the emission angle:
   CALL TRFFCT(RE(I),GSTR(J),GMIN(I),GMAX(I),TRFH,COSNEH)
   DO 40 K = 1, 2
      TRFF(I,J,K) = TRFH(K)
      COSNE(I,J,K) = COSNEH(K)
   40 CONTINUE
30 CONTINUE

C-----------------------------------------------

C- Input of the observed frequency:
   90 READ *,NU
C- If nuo is negative, the program stops:
   IF (NUO.LE.0.0) STOP

C---------- Calculation of the spec. luminosity -----  
C- Set re-integral to 0:
   INTRL(1) = 0.0 DO
   INTRL(2) = 0.0 DO
C- Integration with respect to re:
   DO 60 I = 1, NRE
C- Set g*-integral to 0:
   INTGL(1) = 0.0 DO
   INTGL(2) = 0.0 DO
C- Integration with respect to g*:
   DO 60 J = 1, NG
C- Calculation of the emitted frequency:
   NUE = NUO / G(I,J)
   DO 70 K = 1, 2
C- Calculation of the specific intensity Ispec
C (routine still has to be defined):
    ISPEC(I,J,K) = FCTISP(NUE,RE(I),COSNE(I,J,K))

C- Calculation of the integrand with respect to g*:
    INTG = RE(I)*(2.DO*PI*Q(I,J)*RE(J))**2
    & TRFF(I,J,K)*ISPEC(I,J,K)/
    & SQRT(GSTR(J) - GSTR(J)**2)
    INTGL(K) = INTGL(K) + INTG*WG(J)
70   CONTINUE
60   CONTINUE

C- Calculation of the integrand with respect to re:
    DO 80 K = 1, 2
    INTRL(K) = INTRL(K) + INTGL(K)*WR(I)
80   CONTINUE
50   CONTINUE

C-----------------------------------------------

C--------------- Output of the spec. luminosity ----------------
    LSPEC = INTRL(1) + INTRL(2)
    PRINT *, 'observed frequency: nuo = ', NU0
    PRINT *, 'specific luminosity: Lspec = ', LSPEC
    PRINT *

C-----------------------------------------------

C- Next observed frequency nuo:
    GOTO 90
END

References